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With a Cole–Hopf transformation to solutions of the noncommutative KP hierarchy in terms of Wronski matrices

Aristophanes Dimakis[1] **and Folkert Müller-Hoissen**[2]

[1] Department of Financial and Management Engineering, University of the Aegean, 31 Fostini Str., GR-82100 Chios, Greece
[2] Max-Planck-Institute for Dynamics and Self-Organization, Bunsenstrasse 10, D-37073 Göttingen, Germany

E-mail: dimakis@aegean.gr and folkert.mueller-hoissen@ds.mpg.de

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Abstract
In the case of the KP hierarchy where the dependent variable takes values in an (arbitrary) associative algebra $\mathcal{R}$, it is known that there are solutions which can be expressed in terms of quasideterminants of a Wronski matrix which solves the linear heat hierarchy. We obtain these solutions without the help of quasideterminants in a simple way via solutions of matrix KP hierarchies (over $\mathcal{R}$) and by use of a Cole–Hopf transformation. For this class of exact solutions we work out a correspondence with 'weakly nonassociative' algebras.

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The generalization of the KP equation to the case where the dependent variable takes values in a matrix algebra has already been considered long time ago in [1, 2], for example. The interest in this equation, and more generally in 'soliton equations' where the dependent variable takes values in any associative algebra (see also [3], and in particular [4, 5] for the KP case), is partly due to the fact that there is an elegant way to generate from simple solutions of such a matrix or operator equation complicated solutions of the corresponding scalar equation [6–16]. Moreover, certain developments in string theory motivated the study of soliton equations like the KP equation with the ordinary product of functions replaced by a noncommutative (Groenewold–Moyal) star product (see [17–20] and references cited therein)[3].

[3] In several publications on 'Moyal-deformed' soliton equations, the Moyal-product can be replaced almost completely by *any* associative (noncommutative) product, since the specific properties of the Moyal-product are not actually used. The algebraic properties of such equations are then simply those of (previously studied) matrix versions of these equations. Exceptions are in particular [17–19] where enlarged hierarchies are considered which appear specifically in the Moyal-deformed case. Multi-soliton solutions of the (enlarged) potential KP hierarchy with Moyal-deformed product were obtained in [19] using a method which, in the commutative case, corresponds to the well-known 'trace method' [21] (see also [5], appendix A6, and [22]).

Surprisingly, many integrability features of the scalar KP equation and its hierarchy generalize in some way to the 'noncommutative' version. In [23] (see also [24, 25]) expressions for solutions of the 'noncommutative' potential KP (pKP) hierarchy were found in terms of quasideterminants [26–31], thus achieving a close analogy with classical results for the 'commutative' pKP hierarchy (see [32, 33], for example)[4]. In this communication, we recover these solutions in an elementary way without the use of quasideterminants, via the construction of solutions of matrix pKP hierarchies, where now the matrices have entries in the respective (noncommutative) associative algebra. Moreover, our analysis sheds light on the underlying structure from different perspectives. Section 1 identifies a Cole–Hopf transformation as a basic ingredient. Solutions of the 'noncommutative' pKP hierarchy obtained in this way determine solutions of a certain system of ordinary differential equations. In section 2, we show that conversely solutions of this system determine solutions of the pKP hierarchy. This is achieved by use of results from a very general approach towards solutions of KP hierarchies, developed in [15] (see also [37]). Section 3 explains why Wronski matrices enter the stage under certain familiar additional conditions ('rank 1 condition' and shift operator). Here we make closer contact with the recent work in [25]. Section 4 shows that the associated system of ordinary differential equations can then be cast into the form of matrix Riccati equations and makes contact with the Sato theory [38]. Finally, section 5 contains some further remarks and an appendix draws some consequences for the 'noncommutative' *discrete* KP hierarchy.

1. Cole–Hopf transformation for noncommutative pKP hierarchies and related systems of ordinary differential equations

We recall a result from [16] (see theorem 4.1). Although not explicitly stated there, it holds for elements of an arbitrary associative algebra $\mathcal{A}$ with identity element I, over a field $\mathbb{K}$ of characteristic zero. It is assumed that the elements depend smoothly or as formal power series on independent variables $\mathbf{t} = (t_1, t_2, \ldots)$. Let $\partial : \mathcal{A} \to \mathcal{A}$ be a $\mathbb{K}$-linear derivation which commutes with the partial derivatives ∂_{t_n}.[5]

Proposition 1. *If $X, Y \in \mathcal{A}$ solve the linear heat hierarchy*

$$X_{t_n} = \partial^n(X), \qquad n = 1, 2, 3, \ldots, \tag{1}$$

(where $X_{t_n} = \partial_{t_n}(X)$) and if

$$\partial(X) = RX + QY \tag{2}$$

with constant[6] elements $Q, R \in \mathcal{A}$, then

$$\phi = YX^{-1} \tag{3}$$

solves the pKP hierarchy in the algebra $\mathcal{A}$ with product $A \cdot B = AQB$.

Remark. A functional representation of the pKP hierarchy in $(\mathcal{A}, \cdot)$ is given by $\Omega(\mu) - \Omega(\mu)_{-[\lambda]} = \Omega(\lambda) - \Omega(\lambda)_{-[\mu]}$ with

$$\Omega(\lambda) = \lambda^{-1}(\phi - \phi_{-[\lambda]}) - (\phi - \phi_{-[\lambda]}) \cdot \phi - \phi_{t_1} \tag{4}$$

and the Miwa shift $\phi_{-[\lambda]}(\mathbf{t}) = \phi(\mathbf{t} - [\lambda])$ where $[\lambda] = (\lambda, \lambda^2/2, \lambda^3/3, \ldots)$ (see [16] and also the references cited therein). Expansion in powers of the indeterminates λ, μ generates the pKP hierarchy equations. The pKP hierarchy system is equivalent to

$$\Omega(\lambda) = \vartheta - \vartheta_{-[\lambda]} \tag{5}$$

[4] One link between integrable systems and quasideterminants is given by the fact that 'noncommutative' Darboux transformations [34] can be compactly expressed in the form of a quasideterminant [25, 31, 35, 36].
[5] The reader would not run into problems setting $\partial = \partial_{t_1}$ in the following, because of (1) with $n = 1$.
[6] An element $Q \in \mathcal{A}$ is called *constant* if it does not depend on the variables t_n and satisfies $\partial(Q) = 0$.

with some $\vartheta \in \mathcal{A}$. Under the assumptions of proposition 1, we have $\vartheta = \phi R$ (see the proof of theorem 4.1 in [16]). In this restricted case, the hierarchy equations are thus determined by

$$(\phi - \phi_{-[\lambda]})(\lambda^{-1} - Q\phi - R) - \phi_{t_1} = 0. \tag{6}$$

Equation (3) becomes a *Cole–Hopf* transformation if

$$Y = \partial(X), \tag{7}$$

in which case condition (2) takes the form

$$(I - Q)\partial(X) = RX. \tag{8}$$

Proposition 2. *If $X \in \mathcal{A}$ solves the linear heat hierarchy (1) and (8) with constant $Q, R \in \mathcal{A}$, then*

$$\mathcal{W}(i, j) = \partial^{i+1}(X)X^{-1}R^j \qquad i, j = 0, 1, \ldots \tag{9}$$

satisfy

$$\mathcal{W}(i, j)_{t_n} = \mathcal{W}(i + n, j) - \mathcal{W}(i, j + n) - \sum_{k=0}^{n-1} \mathcal{W}(i, k)Q\mathcal{W}(n - k - 1, j). \tag{10}$$

Proof. By induction (8) leads to

$$\partial^n(X) = R^n X + \sum_{k=0}^{n-1} R^k Q\partial^{n-k}(X) \qquad n = 1, 2, \ldots.$$

With its help and by use of (1) one easily verifies (10). □

Note that $\mathcal{W}(0, 0) = \phi$. Associated with this solution of the pKP hierarchy in the algebra $\mathcal{A}$ with product $A \cdot B = AQB$, we thus have, via (9), a solution $\{\mathcal{W}(i, j)\}$ of the system (10) of ordinary differential equations. In the following section we prove the converse: whenever we have a solution $\{\mathcal{W}(i, j)\}$ of the system (10), then $\mathcal{W}(0, 0)$ solves the pKP hierarchy (in the algebra $\mathcal{A}$ with product $A \cdot B = AQB$).

2. Weakly nonassociative algebras related to pKP solutions

Before recalling a central result from [15] (see also [37]), we need some definitions. An algebra $(\mathbb{A}, \circ)$ (over a commutative ring) is called *weakly nonassociative (WNA)* if it is not associative, but the associator[7] $(a, b \circ c, d)$ vanishes for all $a, b, c, d \in \mathbb{A}$. The middle nucleus $\mathbb{A}' = \{b \in \mathbb{A} | (a, b, c) = 0\ \forall a, c \in \mathbb{A}\}$, which is an *associative* subalgebra, is then also an ideal in $\mathbb{A}$. With respect to an element $\nu \in \mathbb{A} \backslash \mathbb{A}'$ we define the products $a \circ_1 b = a \circ b$ and

$$a \circ_{n+1} b = a \circ (\nu \circ_n b) - (a \circ \nu) \circ_n b \qquad n = 1, 2, \ldots. \tag{11}$$

As a consequence of the WNA condition, these products only depend on the equivalence class $[\nu]$ of ν in $\mathbb{A}/\mathbb{A}'$.

Theorem [15]. *Let $\mathbb{A}$ be any WNA algebra, the elements of which depend smoothly on independent variables $t_1, t_2, \ldots$, and let $\nu \in \mathbb{A} \backslash \mathbb{A}'$ be constant. Then the flows of the system of ordinary differential equations*

$$\phi_{t_n} = -\nu \circ_n \nu + \nu \circ_n \phi + \phi \circ_n \nu - \phi \circ_n \phi \qquad n = 1, 2, \ldots \tag{12}$$

commute and any solution $\phi \in \mathbb{A}'$ solves the pKP hierarchy in $\mathbb{A}'$.

[7] The associator is defined as $(a, b, c) = (a \circ b) \circ c - a \circ (b \circ c)$.

Examples of WNA algebras are obtained as follows [15]. Let $(\mathcal{A}, \cdot)$ be any associative algebra and $L, R : \mathcal{A} \to \mathcal{A}$ linear maps such that

$$[L, R] = 0, \qquad L(a \cdot b) = L(a) \cdot b, \qquad R(a \cdot b) = a \cdot R(b). \tag{13}$$

It is convenient to write La and aR instead of $L(a)$ and $R(a)$. Augmenting $\mathcal{A}$ with a constant element ν and setting

$$\nu \circ \nu = 0, \qquad \nu \circ a = La, \qquad a \circ \nu = -aR, \qquad a \circ b = a \cdot b \tag{14}$$

leads to a WNA algebra $(\mathbb{A}, \circ)$ with $\mathbb{A}' = \mathcal{A}$, provided that there exist $a, b \in \mathcal{A}$ such that $aR \circ b \neq a \circ Lb$. The latter condition ensures that the augmented algebra is *not* associative. As a consequence, we obtain

$$\nu \circ_n a = L^n a, \qquad a \circ_n \nu = -aR^n, \tag{15}$$

and

$$a \circ_n b = \sum_{k=0}^{n-1} aR^k \cdot L^{n-k-1} b. \tag{16}$$

Now (12) reads

$$\phi_{t_n} = L^n \phi - \phi R^n - \sum_{k=0}^{n-1} \phi R^k \cdot L^{n-k-1} \phi \qquad n = 1, 2, \ldots. \tag{17}$$

Introducing

$$\mathcal{W}(i, j) = L^i \phi R^j \qquad i, j = 0, 1, \ldots, \tag{18}$$

and acting on (17) by L^i from the left and by R^j from the right, results in

$$\mathcal{W}(i, j)_{t_n} = \mathcal{W}(i+n, j) - \mathcal{W}(i, j+n) - \sum_{k=0}^{n-1} \mathcal{W}(i, k) \cdot \mathcal{W}(n-k-1, j), \tag{19}$$

assuming that the partial derivatives ∂_{t_n} commute with L and R.

Suppose now that we have a solution $\{\mathcal{W}(i, j)\}$ of the system (19). Then we can choose $\mathcal{A}$ as the associative algebra generated by this set (with product $\cdot$) and define the maps L and R via (18) and (13). It follows (by use of the theorem) that $\phi = \mathcal{W}(0, 0)$ solves the pKP hierarchy in $(\mathcal{A}, \cdot)$.

In fact, in the first section we have shown how a subclass of solutions to the pKP hierarchy determines solutions $\{\mathcal{W}(i, j)\}$ of (19) in the case where the product in $\mathcal{A}$ is given by $A \cdot B = AQB$ with a constant element $Q \in \mathcal{A}$.

In particular, we have seen that the system (10) is a special case of the hierarchy (12) of ordinary differential equations in a WNA algebra $\mathbb{A}$, which according to the above theorem determines solutions of the pKP hierarchy in $\mathbb{A}'$. An example of (10) appeared in [25] and this will be the subject of the next section.

3. Solutions of noncommutative pKP hierarchies in terms of Wronski matrices

Let us now choose $\mathcal{A}$ as the algebra of $N \times N$ matrices with entries in a unital associative algebra $\mathcal{R}$ and product $A \cdot B = AQB$ with a constant $N \times N$ matrix Q. Let $\mathbf{e}_k$ be the N-component vector with all entries zero except for the identity element in the kth row. Choosing the rank 1 matrix

$$Q = \mathbf{e}_N \mathbf{e}_N^T \tag{20}$$

(where ^{T} means taking the transpose), any solution ϕ of the pKP hierarchy in $(\mathcal{A}, \cdot)$ determines via

$$\varphi = \mathbf{e}_N^T \phi \mathbf{e}_N \tag{21}$$

a solution of the pKP hierarchy in $\mathcal{R}$. Choosing moreover

$$R = \Lambda = \sum_{k=1}^{N-1} \mathbf{e}_k \mathbf{e}_{k+1}^T, \tag{22}$$

which is the 'left shift' ($\Lambda \mathbf{e}_1 = 0$ and $\Lambda \mathbf{e}_k = \mathbf{e}_{k-1}$ for $k = 2, \ldots, N$), condition (8) becomes

$$\left(I - e_N e_N^T\right)\partial(X) = \Lambda X. \tag{23}$$

This tells us that X is a Wronski matrix, i.e.

$$X = W(\Theta) = \begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_N \\ \partial(\theta_1) & \partial(\theta_2) & \cdots & \partial(\theta_N) \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{N-1}(\theta_1) & \partial^{N-1}(\theta_2) & \cdots & \partial^{N-1}(\theta_N) \end{pmatrix}, \tag{24}$$

with a row vector $\Theta = (\theta_1, \ldots, \theta_N)$ of elements of $\mathcal{R}$. We simply write W instead of $W(\Theta)$ in the following. The next result is an immediate consequence of proposition 1.

Proposition 3. *If $\theta_1, \ldots, \theta_N$ solve the linear heat hierarchy, i.e. $\Theta_{t_n} = \partial^n(\Theta)$, $n = 1, 2, \ldots,$ and if the Wronski matrix W is invertible, then*

$$\phi = \partial(W)W^{-1} \tag{25}$$

solves the pKP hierarchy in the algebra of $N \times N$ matrices with entries in $\mathcal{R}$ and product $A \cdot B = AQB$ with Q defined in (20). Furthermore, Φ defined in (21) then solves the pKP hierarchy in $\mathcal{R}$.

According to section 1, ϕ given by (25) determines a solution of the system (10). As a consequence,

$$\mathcal{Q}(i, j) = -\mathbf{e}_N^T \mathcal{W}(i, j)\mathbf{e}_N \qquad i, j = 0, 1, \ldots \tag{26}$$

solve the system

$$\mathcal{Q}(i, j)_{t_n} = \mathcal{Q}(i + n, j) - \mathcal{Q}(i, j + n) + \sum_{k=0}^{n-1} \mathcal{Q}(i, k)\mathcal{Q}(n - k - 1, j), \tag{27}$$

which appeared in [25]. From the above definition, we immediately find the following expression in terms of quasideterminants:

$$\mathcal{Q}(i, j) = -\mathbf{e}_N^T \partial^{i+1}(W)W^{-1}\mathbf{e}_{N-j} = \begin{vmatrix} W & \mathbf{e}_{N-j} \\ \mathbf{e}_N^T \partial^{i+1}(W) & \boxed{0} \end{vmatrix} \tag{28}$$

(see [28, 29, 31] for the notation). The authors of [25] used Darboux transformations and properties of quasideterminants to derive these results. Knowing that (27) holds, we can also refer directly to the arguments of section 2 (instead of referring to the matrix solution ϕ) to conclude that

$$\varphi = -\mathcal{Q}(0, 0) = -\mathbf{e}_N^T \partial(W)W^{-1}\mathbf{e}_N = \begin{vmatrix} W & \mathbf{e}_N \\ \mathbf{e}_N^T \partial(W) & \boxed{0} \end{vmatrix} \tag{29}$$

solves the pKP hierarchy in $\mathcal{R}$.[8]

Example. For $N = 2$ we have

$$W = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_1' & \theta_2' \end{pmatrix}, \qquad W^{-1} = \begin{pmatrix} \left(\theta_1 - \theta_2 \theta_2'^{-1} \theta_1'\right)^{-1} & \left(\theta_1' - \theta_2' \theta_2^{-1} \theta_1\right)^{-1} \\ \left(\theta_2 - \theta_1 \theta_1'^{-1} \theta_2'\right)^{-1} & \left(\theta_2' - \theta_1' \theta_1^{-1} \theta_2\right)^{-1} \end{pmatrix}, \tag{30}$$

where $\theta_k' = \partial(\theta_k)$, and we need to assume that the inverses exist. This leads to

$$\phi = \begin{pmatrix} 0 & 1 \\ \left(\theta_1'' - \theta_2''(\theta_2')^{-1}\theta_1'\right)\left(\theta_1 - \theta_2(\theta_2')^{-1}\theta_1'\right)^{-1} & \varphi \end{pmatrix}, \tag{31}$$

where 1 stands for the identity element of $\mathcal{R}$ and

$$\varphi = \left(\theta_1'' - \theta_2'' \theta_2^{-1} \theta_1\right)\left(\theta_1' - \theta_2' \theta_2^{-1} \theta_1\right)^{-1}. \tag{32}$$

Particular solutions of the heat hierarchy are[9]

$$\theta_k = \sum_{j=1}^{M} A_{k,j} e^{\xi(\alpha_j)} B_{k,j} \qquad k = 1, \ldots, N, \tag{33}$$

with some $M \in \mathbb{N}$, constant elements $A_{k,j}, B_{k,j}, \alpha_j \in \mathcal{R}$ and $\xi(\alpha) = \sum_{m \geqslant 1} t_m \alpha^m$. In the 'commutative case', one recovers N-soliton solutions in this way [33].

4. Linearization of the system (27)

Let us introduce the infinite matrix $\mathcal{Q} = (\mathcal{Q}(i, j))$, and the corresponding shift operator

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & & \ddots \end{pmatrix}. \tag{34}$$

Then (27) can be expressed as a system of matrix Riccati equations,

$$\mathcal{Q}_{t_n} = \Lambda^n \mathcal{Q} - \mathcal{Q}(\Lambda^T)^n + \mathcal{Q} P_n \mathcal{Q} \qquad n = 1, 2, \ldots, \tag{35}$$

with $P_1 = \mathbf{e}_1 \mathbf{e}_1^T$, where now $\mathbf{e}_1^T = (1, 0, \ldots)$, and

$$P_n = \sum_{k=0}^{n-1} (\Lambda^T)^k P_1 \Lambda^{n-k-1} \qquad n = 2, 3, \ldots. \tag{36}$$

Such matrix Riccati equations are well known to be linearizable. The corresponding linear system is

$$\begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}_{t_n} = \begin{pmatrix} \Lambda^T & P_1 \\ 0 & \Lambda \end{pmatrix}^n \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix} \qquad n = 1, 2, \ldots, \tag{37}$$

which determines a solution of the matrix Riccati system via $\mathcal{Q} = \mathcal{Y}\mathcal{X}^{-1}$. Let us introduce

$$\hat{\mathcal{X}}_m = \begin{cases} \mathcal{X}_{-m} & m < 0 \\ \mathcal{Y}_{m+1} & m \geqslant 0, \end{cases} \tag{38}$$

[8] This also solves the technical problems met by the authors of [25] in their 'direct approach'. For the use of computer algebra to perform computations of the kind considered in section 5 of [25], see also the appendix of [37].

[9] The exponentials are at least well defined as formal power series in $\mathbf{t}$. Other solutions of the heat hierarchy are given by linear combinations of Schur polynomials in $\mathbf{t}$ with constant coefficients in $\mathcal{R}$.

where $\mathcal{X}_m, \mathcal{Y}_m, m = 1, 2, \ldots$, are the rows of the matrices $\mathcal{X}, \mathcal{Y}$ (with entries in $\mathcal{R}$). In terms of the vector $\hat{\mathcal{X}} = (\hat{\mathcal{X}}_m)_{m\in\mathbb{Z}}$, this linear system takes the simple form

$$\hat{\mathcal{X}}_{t_n} = \hat{\Lambda}^n \hat{\mathcal{X}} \qquad n = 1, 2, \ldots, \tag{39}$$

with the two-sided infinite shift matrix $\hat{\Lambda}$. The solutions are given by $\hat{\mathcal{X}}(\mathbf{t}) = e^{\xi(\hat{\Lambda})}\hat{\mathcal{X}}(0)$ with $\xi(\hat{\Lambda}) = \sum_{n\geqslant 1} t_n \hat{\Lambda}^n$. All this makes contact with Sato's formulation of the KP hierarchy as flows on an infinite-dimensional Grassmann manifold (see in particular [38–40]), but here the components of $\hat{\mathcal{X}}$ are taken from the (typically noncommutative) associative algebra $\mathcal{R}$.

5. Further remarks

In the transition from (17) to (19) the linear maps L and R get hidden away and the infinite-dimensional shift operator enters the stage. This is made explicit in section 4, under the restrictions imposed in section 3. A system of the form (19), respectively (27), already appeared in [41] (p 186) and in [40] (see in particular p 29) as a description of the (ordinary) KP hierarchy as Sato flows on the infinite (universal) Grassmann manifold (by vector fields associated with powers $\hat{\Lambda}^n, n = 1, 2, \ldots$, of the shift operator), see also [42]. Later it reappeared in [43, 44] as the 'Sato system' of the (ordinary) KP hierarchy[10]. From a practical point of view, in particular when addressing exact solutions, it is in our opinion not of much help and it is more convenient and simpler to deal with (17) (or the more general system (12), see also [15, 37]). From a theoretical point of view, we have seen that the system (19) indeed has its merits. In particular, it helped us bridging different approaches to solving a (noncommutative) KP hierarchy.

Our work makes evident that the choice of the shift operator ($R = \Lambda$ in section 3) is rather special and there are others (see also [37]). We refer to the interesting discussion in [45] concerning the role of the shift operator in Sato theory and corresponding alternatives.

Appendix. A note on the (noncommutative) discrete pKP hierarchy

The (noncommutative) potential discrete KP (pDKP) hierarchy in an associative algebra $(\mathcal{A}, \cdot)$ (see [37, 46] and the references cited therein) is given by

$$\hat{\Omega}(\mu)^+ - \hat{\Omega}(\mu)_{-[\lambda]} = \hat{\Omega}(\lambda)^+ - \hat{\Omega}(\lambda)_{-[\mu]}, \tag{A.1}$$

where

$$\hat{\Omega}(\lambda) = \lambda^{-1}(\phi - \phi_{-[\lambda]}) - (\phi^+ - \phi_{-[\lambda]}) \cdot \phi, \tag{A.2}$$

and $\phi = (\phi_k)_{k\in\mathbb{Z}}, \phi_k^+ = \phi_{k+1}$, with $\phi_k \in \mathcal{A}$. (A.1) is equivalent to

$$\hat{\Omega}(\lambda) = \vartheta^+ - \vartheta_{-[\lambda]} \tag{A.3}$$

with some $\vartheta = (\vartheta_k)_{k\in\mathbb{Z}}, \vartheta_k \in \mathcal{A}$. Taking the limit $\lambda \to 0$ in (A.3) results in

$$\phi_{t_1} = (\phi^+ - \phi) \cdot \phi + \vartheta^+ - \vartheta, \tag{A.4}$$

by use of which (A.3) is turned into the pKP system (5). Under the assumptions of proposition 1, we have $\vartheta = \phi R$ (see the remark in section 1), so that (A.4) takes the form

$$\phi_{t_1} - (\phi^+ - \phi)(Q\phi + R) = 0. \tag{A.5}$$

[10] See equation (7.1) in [43] and (2.8) in [44]. The correspondence is given by $\mathcal{W}(i, j) \mapsto -W^i_{j+1}$, respectively $\mathcal{Q}(i, j) \mapsto W^i_{j+1}$. The linearization presented in section 4 also appeared in these papers (where $\mathcal{R}$ is the commutative algebra of functions of $\mathbf{t}$). We believe that our presentation is somewhat improved.

Assuming furthermore the Cole–Hopf restriction (7), we have

$$\phi_{t_1} = \partial^2(X)X^{-1} - \phi^2 = \partial^2(X)X^{-1} - \phi Q\phi - \phi(I - Q)\phi. \tag{A.6}$$

With the help of (8), which is $(I - Q)\phi = R$, this becomes $\phi_{t_1} = \partial^2(X)X^{-1} - \phi(Q\phi + R)$. Inserting this expression into (A.4) leads to $\partial^2(X)X^{-1} = \phi^+(Q\phi + R) = \phi^+\partial(X)X^{-1}$, which is

$$\phi^+ = \partial^2(X)(\partial(X))^{-1}. \tag{A.7}$$

Hence, any invertible solution X of the linear heat hierarchy, subject to (8) (with any constant Q, R), determines a solution

$$\phi_k = \partial^{k+1}(X)(\partial^k(X))^{-1} \tag{A.8}$$

of the pDKP hierarchy, restricted to non-negative integers k, provided that the inverse of $\partial^k(X)$ exists for all k. If the inverse ∂^{-1} of ∂ and its powers can be defined on X, this extends to the whole lattice. In particular, 'Wronski solutions' of pKP hierarchies as considered in section 3 extend in this way to solutions of the corresponding pDKP hierarchies.

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